

A CRITERION FOR ESSENTIAL SELF-ADJOINTNESS OF A SYMMETRIC OPERATOR DEFINED BY SOME INFINITE HERMITIAN MATRIX WITH UNBOUNDED ENTRIES

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ABSTRACT. We shall consider a double infinite, hermitian, complex entry matrix $A = [a_{x,y}]_{x,y \in \mathbb{Z}}$, with $a_{x,y}^* = a_{y,x}$, $x, y \in \mathbb{Z}$. Assuming that the matrix is *almost of a finite bandwidth*, i.e. there exists an integer $n > 0$ and exponent $\gamma \in [0, 1)$ such that $a_{x,x+z} = 0$ for all $z > n\langle x \rangle^\gamma$ and the growth of the ℓ_1 norm of a row is slower than $|x|^{1-\gamma}$ for $|x| \gg 1$, i.e. $\lim_{|x| \rightarrow +\infty} |x|^{\gamma-1} \sum_y |a_{xy}| = 0$ we prove that the corresponding symmetric operator, defined on compactly supported sequences, is essentially self-adjoint in $\ell_2(\mathbb{Z})$. In the case $\gamma = 0$ (the so called (nJ) -matrices) we prove that there exists $c_* > 0$, depending only on n , such that the condition $\limsup_{|x| \rightarrow +\infty} |x|^{-1} \sum_y |a_{xy}| \leq c_*$ suffices to conclude essential self-adjointness.

1. INTRODUCTION

We shall consider a double infinite, hermitian, complex entry matrix $A = [a_{x,y}]_{x,y \in \mathbb{Z}}$, with $a_{x,y}^* = a_{y,x}$, $x, y \in \mathbb{Z}$. Here a^* denotes the complex conjugate of $a \in \mathbb{C}$. We assume furthermore that the matrix is *almost of a finite bandwidth*, i.e. there exists an integer $n \geq 1$ and exponent $\gamma \in [0, 1)$ such that

$$a_{x,x+z} = 0 \text{ for all } z > n\langle x \rangle^\gamma. \quad (1.1)$$

Here, for given a we let $\langle a \rangle := (1 + |a|^2)^{1/2}$. With the help of matrix A we can define a symmetric operator on the subset $c_0(\mathbb{Z})$ of the complex Hilbert space $\ell_2(\mathbb{Z})$ – the space consisting of all double infinite sequences $f = (f_x)$ equipped with the norm

$$\|f\|_{\ell_2(\mathbb{Z})} := \left\{ \sum_x |f_x|^2 \right\}^{1/2} < +\infty.$$

Here $c_0(\mathbb{Z})$ is the subspace containing all compactly supported f . The operator is given by

$$(Af)_x := \sum_y a_{xy} f_y, \quad x \in \mathbb{Z}, \quad f \in c_0(\mathbb{Z}). \quad (1.2)$$

According to Theorem 4, p. 102 of [1], assumption (1.1) implies that the operator is closable. Denote its closure by $\bar{A} : D(\bar{A}) \rightarrow \ell_2(\mathbb{Z})$. In our principal result, see Theorem 2.1 below, we formulate a sufficient condition, in terms of the growth of $|a_{xy}|$, see (2.4) below, for the

operator \bar{A} to be self-adjoint. The above means that the deficiency index of $A : c_0(\mathbb{Z}) \rightarrow \ell_2(\mathbb{Z})$ equals $(0, 0)$, see [10].

Note that in the particular case when $\gamma = 0$ we have

$$a_{x,y} = 0 \text{ for all } |x - y| > n \quad (1.3)$$

and the definition coincides with the usual definition of (nJ) -matrices, see [13], (sometimes also called *finite bandwidth* matrices). When $n = 1$ they are called *Jacobi* matrices and play an important role in the theory of Hamburger moment problem. This case has been well studied in the literature, see e.g. [5, 6, 7, 8, 12] and the references contained therein, although also then our results formulated in Corollary 2.2 and Theorem 2.3 below seem to be new.

2. THE STATEMENT OF THE MAIN RESULT

Since matrix $A = [a_{x,y}]$ is hermitian the operator \bar{A} is obviously symmetric, i.e.

$$\langle g, \bar{A}f \rangle_{\ell_2(\mathbb{Z})} = \langle \bar{A}g, f \rangle_{\ell_2(\mathbb{Z})}, \quad f, g \in D(\bar{A}). \quad (2.1)$$

Here, $\langle \cdot, \cdot \rangle_{\ell_2(\mathbb{Z})}$ denotes the usual scalar product in $\ell_2(\mathbb{Z})$. Let $f \in \ell_2(\mathbb{Z})$ be such that the functional

$$\varphi(g) := \langle \bar{A}g, f \rangle_{\ell_2(\mathbb{Z})}, \quad g \in D(\bar{A}) \quad (2.2)$$

is bounded, i.e. for some $C > 0$

$$|\varphi(g)| \leq C \|g\|_{\ell_2}, \quad g \in D(\bar{A}). \quad (2.3)$$

Self-adjointness of \bar{A} means that any f , for which (2.3) holds, belongs to $D(\bar{A})$ and, as a consequence, (2.1) is in force.

For example, if there exists $M > 0$ such that $\sum_y |a_{xy}| \leq M$ for all $x \in \mathbb{Z}$ then \bar{A} is bounded on $\ell_2(\mathbb{Z})$, see Example III.2.3, p. 143 of [3], therefore it is self-adjoint. Our main result can be stated as follows.

Theorem 2.1. *Suppose that for some $\gamma \in [0, 1)$ the entries of matrix A satisfy both condition (1.1) and*

$$\lim_{|x| \rightarrow +\infty} \frac{1}{\langle x \rangle^{1-\gamma}} \left(\sum_y |a_{xy}| \right) = 0. \quad (2.4)$$

Then, operator A , given by (1.2), is essentially selfadjoint on $\ell_2(\mathbb{Z})$.

Using the theorem for $\gamma = 0$ we immediately conclude the following.

Corollary 2.2. *The conclusion of Theorem 2.1 holds when A is a hermitian, nJ -matrix (i.e. (1.3) is in force) whose entries satisfy*

$$\lim_{|x| \rightarrow +\infty} \frac{1}{\langle x \rangle} \left(\sum_y |a_{xy}| \right) = 0. \quad (2.5)$$

In fact, in the case of nJ -matrices, one can show a little stronger result, relaxing a bit assumption (2.5).

Theorem 2.3. *There exists $c_* > 0$ depending only on n such that the conclusion of Theorem 2.1 holds for any hermitian nJ -matrix $A = [a_{xy}]$ that satisfies*

$$\limsup_{|x| \rightarrow +\infty} \frac{1}{\langle x \rangle} \left(\sum_y |a_{xy}| \right) \leq c_*. \quad (2.6)$$

Example. The condition (2.6) is in some sense optimal. Suppose that $\delta > 1$ is arbitrary. Consider the Jacobi matrix with entries given by

$$a_{x,x+z} = \begin{cases} 0, & \text{if } x \leq 0, \text{ or } x+z \leq 0, \text{ or } z = 0, \text{ or } z > 2, \\ x^\delta, & \text{if } x > 0 \text{ and } z = 1. \end{cases}$$

According to Corollary 1, p. 267, of [5] the index of deficiency of the respective operator $A : c_0(\mathbb{Z}) \rightarrow \ell_2(\mathbb{Z})$ equals then $(1, 1)$. Therefore A cannot be essentially self-adjoint, see Corollary 2.2 of [12].

3. PROOF OF THEOREM 2.1

Recall the classical criterion for the essential self-adjointness of a symmetric operator, see Theorem 3 of Section 33.2 of [9], or Corollary 2.2 of [12]. Adjusted to our settings it reads as follows: suppose that a closed operator \bar{A} is symmetric and

$$R(I - i\bar{A}) = \ell_2(\mathbb{Z}) = R(I + i\bar{A}). \quad (3.1)$$

Then, it is self-adjoint.

To prove (3.1) we show that for any $g = (g_x) \in c_0(\mathbb{Z})$ there exists $f = (f_x)$ such that

$$f_x - \sum_y a_{xy} f_y = g_x, \quad x \in \mathbb{Z}, \quad (3.2)$$

$$\sum_x \langle x \rangle^{2k} |f_x|^2 < +\infty, \quad \forall k = 1, 2, \dots$$

Observe that the infinite summation range appearing in the first equation is in fact finite. Indeed, from (1.1) and symmetry it follows that

$$a_{x,x-z} = 0 \text{ for all } z > c_n \langle x \rangle^\gamma, \quad (3.3)$$

where

$$c_n := \max \left\{ 2n, 2^{(1+\gamma/2)/(1-\gamma)} n^{1/(1-\gamma)} \right\}. \quad (3.4)$$

Combining this with (1.1) we conclude that

$$a_{x,x+z} = 0 \text{ for all } |z| > c_n \langle x \rangle^\gamma. \quad (3.5)$$

Furthermore, note that any f satisfying conditions (3.2) belongs to $D(\bar{A})$. Indeed, consider $f_x^{(N)} := f_x 1_{|x| \leq N}$. Thanks to the second condition of (3.2) and (2.4) we can easily argue that $(I - iA)f^{(N)} \rightarrow g$ and $f^{(N)} \rightarrow f$, strongly in $\ell_2(\mathbb{Z})$, as $N \rightarrow +\infty$. Since \bar{A} is the closure of A we conclude that $f \in D(\bar{A})$ and $(I - i\bar{A})f = g$. Due to the fact that $c_0(\mathbb{Z})$ is dense in $\ell_2(\mathbb{Z})$ and that the range $R(I - i\bar{A})$ is closed we conclude that $R(I - i\bar{A}) = \ell_2(\mathbb{Z})$. The proof of the second equality in (3.1) goes along the same lines. What yet remains to be shown is therefore (3.2).

Proof of (3.2). Let

$$\chi_N(r) := \begin{cases} r, & |r| \leq N, \\ N, & r \geq N, \\ -N, & r \leq -N. \end{cases}$$

For a fixed integer N define $A^{(N)}$ as a bounded, symmetric operator corresponding to the hermitian matrix whose entries equal

$$a_{xy}^{(N)} := \chi_N(a_{xy}) 1_{|x-y| \leq N}, \quad x, y \in \mathbb{Z}.$$

Given $g \in c_0(\mathbb{Z})$ there is a (unique) $\tilde{f}^{(N)} \in \ell_2(\mathbb{Z})$ such that

$$(I - iA^{(N)})\tilde{f}^{(N)} = g. \quad (3.6)$$

We show that for any positive integer k there exists a constant $C > 0$ such that

$$\sum_x \langle x \rangle^{2k} |\tilde{f}_x^{(N)}|^2 \leq C, \quad N \geq 1. \quad (3.7)$$

Taking this claim for granted (its proof shall be shown momentarily) we finish the proof of (3.2). Using condition (3.7) with any $k > 0$ we conclude that the tails of the infinite sums defining the $\ell_2(\mathbb{Z})$ norms of $(\tilde{f}^{(N)})$ are uniformly small in N . This proves that the sequence is strongly precompact in $\ell_2(\mathbb{Z})$, see e.g. Theorem 4.20.1 of [2]. In fact, observe that each $\tilde{f}^{(N)} \in D(\bar{A})$. Indeed, let $\tilde{f}^{(N,M)} := (\tilde{f}_x^{(N)} 1_{|x| \leq M})$ for an integer $M \geq 1$. Of course $\tilde{f}^{(N,M)} \in c_0(\mathbb{Z}) \subset D(\bar{A})$, and it converges

to $\tilde{f}^{(N)}$ strongly in $\ell_2(\mathbb{Z})$, as $M \rightarrow +\infty$. On the other hand, from (3.7) for any $k, \tilde{c} > 0$ there exists $C > 0$ such that

$$\sup_{|y| \geq \tilde{c}|x|} |\tilde{f}_y^{(N)}|^2 \leq \frac{C}{\langle x \rangle^{2k+4}}, \quad N \geq 1, x \in \mathbb{Z}. \quad (3.8)$$

Using (3.5), we can estimate

$$\begin{aligned} \sum_x \langle x \rangle^{2k} |(A\tilde{f}^{(N,M)})_x|^2 &\leq \sum_x \langle x \rangle^{2k} \left(\sum_{|y-x| \leq c_n \langle x \rangle^\gamma} |a_{xy}| |\tilde{f}_y^{(N)}| \right)^2 \\ &\leq \sum_x \langle x \rangle^{2k} \sup_{|y-x| \leq c_n \langle x \rangle^\gamma} |\tilde{f}_y^{(N)}|^2 \left(\sum_y |a_{xy}| \right)^2. \end{aligned} \quad (3.9)$$

Since $\gamma \in [0, 1)$ condition $|y - x| \leq c_n \langle x \rangle^\gamma$ implies that there exists $\tilde{c} > 0$ such that $|y| \geq \tilde{c}|x|$ for all $x, y \in \mathbb{Z}$. Thanks to (3.8) the utmost right hand side of (3.9) can be estimated then by

$$\sum_x \langle x \rangle^{2k} \sup_{|y| \geq \tilde{c}|x|} |\tilde{f}_y^{(N)}|^2 \left(\sum_y |a_{xy}| \right)^2 \leq C \sum_x \langle x \rangle^{-4} \left(\sum_y |a_{xy}| \right)^2. \quad (3.10)$$

This together with (2.4) imply that there exists $C_1 > 0$ such that

$$\sum_x \langle x \rangle^{2k} |(A\tilde{f}^{(N,M)})_x|^2 \leq C_1 \sum_x \langle x \rangle^{-2-2\gamma}, \quad N, M \geq 1. \quad (3.11)$$

In consequence $(A\tilde{f}^{(N,M)})$, $M \geq 1$ is strongly precompact in $\ell_2(\mathbb{Z})$, for a fixed N , and since \bar{A} is the closure of A we obtain $\tilde{f}^{(N)} \in D(\bar{A})$ and

$$\bar{A}\tilde{f}^{(N)} = \lim_{M \rightarrow +\infty} A\tilde{f}^{(N,M)}.$$

In addition, we also infer that

$$(\bar{A}\tilde{f}^{(N)})_x = \sum_y a_{xy} \tilde{f}_y^{(N)}, \quad x \in \mathbb{Z} \quad (3.12)$$

and that for any $k > 0$ there exists a constant $C > 0$

$$\sum_x \langle x \rangle^{2k} |(\bar{A}\tilde{f}^{(N)})_x|^2 \leq C, \quad N \geq 1. \quad (3.13)$$

From (3.7) and (3.13) we conclude that both sequences $(\tilde{f}^{(N)})$ and $(\bar{A}\tilde{f}^{(N)})$ are strongly precompact in $\ell_2(\mathbb{Z})$. Choosing a suitable subsequences if necessary we can assume with no loss of generality that $\tilde{f}^{(N)} \rightarrow f$ and $\bar{A}\tilde{f}^{(N)}$ converges to some h strongly in $\ell_2(\mathbb{Z})$, as $N \rightarrow$

$+\infty$. Then $f \in D(\bar{A})$ and $\bar{A}f = h$. In light of (3.6), to finish the proof of (3.2) it suffices to show that

$$\lim_{N \rightarrow +\infty} \|A^{(N)} \tilde{f}^{(N)} - \bar{A} \tilde{f}^{(N)}\|_{\ell_2(\mathbb{Z})} = 0. \quad (3.14)$$

Estimating as in (3.9) and (3.10) we obtain

$$\begin{aligned} & \sum_x |((\bar{A} - A^{(N)}) \tilde{f}^{(N)})_x|^2 \\ & \leq \sum_x \sup_{|y| \geq \tilde{c}|x|} |\tilde{f}_y^{(N)}|^2 \left(\sum_y |a_{xy} - a_{xy}^{(N)}| \right)^2 \\ & \leq C \sum_x \langle x \rangle^{-2k-4} \left(\sum_y |a_{xy} - a_{xy}^{(N)}| \right)^2 \rightarrow 0, \end{aligned}$$

as $N \rightarrow +\infty$. The passage to the limit on the utmost right hand side can be argued easily by virtue of the Lebesgue dominated convergence theorem. This ends the proof of (3.2), modulo the fact that estimate (3.7) still requires to be shown. Its proof is an adaptation to the present case of an argument used in [11], see also Section 2.7.4 of [4]. Define a bounded operator $T : \ell_2(\mathbb{Z}) \rightarrow \ell_2(\mathbb{Z})$ by the formula $Tf_x := t_x f_x$, where

$$t_x := \langle X \rangle^k 1_{|x| < X} + \langle x \rangle^k 1_{X \leq |x| \leq Y} + \langle Y \rangle^k 1_{Y < |x|}, \quad x \in \mathbb{Z} \quad (3.15)$$

and $0 < X < Y$ are some constants to be determined later on. Directly from (3.15) it follows that

$$|t_x - t_y| \leq |\langle x \rangle^k - \langle y \rangle^k|, \quad \forall x, y \in \mathbb{Z}. \quad (3.16)$$

Applying T to both sides of (3.6) and taking inner product against $T\tilde{f}^{(N)}$ on both sides of the aforementioned equation we conclude that

$$\begin{aligned} & \|T\tilde{f}^{(N)}\|_{\ell_2(\mathbb{Z})}^2 + i \langle T\tilde{f}^{(N)}, [T, A^{(N)}] \tilde{f}^{(N)} \rangle_{\ell_2(\mathbb{Z})} + i \langle T\tilde{f}^{(N)}, A^{(N)} T\tilde{f}^{(N)} \rangle_{\ell_2(\mathbb{Z})} \\ & = \langle T\tilde{f}^{(N)}, Tg \rangle_{\ell_2(\mathbb{Z})}, \end{aligned} \quad (3.17)$$

where $[T, A^{(N)}] := TA^{(N)} - A^{(N)}T$ is the commutator of T and $A^{(N)}$. Thanks to symmetry of $A^{(N)}$ we have

$$\operatorname{Re} i \langle T\tilde{f}^{(N)}, A^{(N)} T\tilde{f}^{(N)} \rangle_{\ell_2(\mathbb{Z})} = 0.$$

Here $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of a complex number z . Taking the real part of the expressions appearing on both sides of (3.17) we obtain

$$\begin{aligned} & \|T\tilde{f}^{(N)}\|_{\ell_2(\mathbb{Z})}^2 - \operatorname{Im} \langle T\tilde{f}^{(N)}, [T, A^{(N)}] \tilde{f}^{(N)} \rangle_{\ell_2(\mathbb{Z})} \\ & = \operatorname{Re} \langle T\tilde{f}^{(N)}, Tg \rangle_{\ell_2(\mathbb{Z})}. \end{aligned} \quad (3.18)$$

Note that

$$[T, A^{(N)}]\tilde{f}_x^{(N)} = \sum_y a_{xy}^{(N)}(t_x - t_y)f_y^{(N)},$$

therefore

$$\langle T\tilde{f}^{(N)}, [T, A^{(N)}]\tilde{f}^{(N)} \rangle_{\ell_2(\mathbb{Z})} = \sum_x \sum_y a_{yx}^{(N)}(t_x - t_y)t_x \tilde{f}_x^{(N)}(\tilde{f}_y^{(N)})^*.$$

The above expression can be bounded as follows

$$|\langle T\tilde{f}^{(N)}, [T, A^{(N)}]\tilde{f}^{(N)} \rangle_{\ell_2(\mathbb{Z})}| \leq \sum_x \sum_y |a_{xy}|t_x|t_y - t_x||\tilde{f}_x^{(N)}||\tilde{f}_y^{(N)}|. \quad (3.19)$$

Applying Young's inequality we can estimate the right hand side of (3.19) by $I_1 + I_2$, where

$$I_1 := \frac{1}{2} \sum_x \sum_y |a_{xy}|t_x|t_y - t_x||\tilde{f}_x^{(N)}|^2, \quad (3.20)$$

$$I_2 := \frac{1}{2} \sum_y \sum_x |a_{xy}|t_x|t_y - t_x||\tilde{f}_y^{(N)}|^2.$$

Using condition (3.5) and the fact that t_x is constant for $|x| \leq X$, or $Y \leq |x|$ we conclude that

$$I_1 \leq \frac{1}{2} \sum_{x=\bar{X}}^{\bar{Y}} \sum_{y=x-c_n\langle x \rangle^\gamma}^{x+c_n\langle x \rangle^\gamma} |a_{xy}|t_x|t_x - t_y||\tilde{f}_x^{(N)}|^2,$$

where $\bar{X} := X - c_n\langle X \rangle^\gamma$ and $\bar{Y} := Y + c_n\langle Y \rangle^\gamma$. Thanks to (3.16) we can estimate

$$I_1 \leq \frac{1}{2} \sum_{x=\bar{X}}^{\bar{Y}} t_x^2 |\tilde{f}_x^{(N)}|^2 \left\{ \frac{\langle x \rangle^k}{t_x} \sum_{y=x-c_n\langle x \rangle^\gamma}^{x+c_n\langle x \rangle^\gamma} |a_{xy}| \left| \frac{\langle y \rangle^k}{\langle x \rangle^k} - 1 \right| \right\}. \quad (3.21)$$

Observe that

$$\frac{\langle x \rangle^k}{t_x} \leq (c_n + 1)^{k/2}, \quad \text{for } \bar{X} \leq x \leq \bar{Y}. \quad (3.22)$$

Choose an arbitrary $\delta \in (0, 1)$. Note that for $|m| \leq n\langle x \rangle^\gamma$ there exist constants $C, C' > 0$ such that

$$\left| \left(\frac{1 + (x+m)^2}{1+x^2} \right)^{k/2} - 1 \right| \leq C \frac{|xm| + m^2/2}{1+x^2} \leq C'(\langle x \rangle^{\gamma-1} + \langle x \rangle^{2\gamma-2}) \quad (3.23)$$

for all x . Combining (3.23) with (2.4) we conclude that for any $\delta \in (0, 1)$ there exists X , depending on parameters n, δ, k, γ , such that

$$\sum_{y=x-c_n\langle x \rangle^\gamma}^{x+c_n\langle x \rangle^\gamma} |a_{xy}| \left| \frac{\langle y \rangle^k}{\langle x \rangle^k} - 1 \right| < \frac{\delta}{(c_n + 1)^{k/2}}, \quad \text{for } |x| \geq \bar{X}. \quad (3.24)$$

This together with (3.22) imply that

$$I_1 \leq \frac{\delta}{2} \|T\tilde{f}^{(N)}\|_{\ell_2(\mathbb{Z})}^2. \quad (3.25)$$

Likewise

$$I_2 \leq \frac{1}{2} \sum_{y=\bar{X}}^{\bar{Y}} t_y^2 |\tilde{f}_y^{(N)}|^2 \sum_{x=y-c_n\langle y \rangle^\gamma}^{y+c_n\langle y \rangle^\gamma} \frac{t_x}{t_y^2} |a_{xy}| |\langle x \rangle^k - \langle y \rangle^k|.$$

Since

$$\frac{1}{C_*} \leq \frac{t_x}{\langle x \rangle^k} \leq C_*, \quad x = \bar{X} - c_n\langle \bar{X} \rangle^\gamma, \dots, \bar{Y} + c_n\langle \bar{Y} \rangle^\gamma$$

for some constant $C_* > 0$ that depends only on n, γ, k , we conclude that

$$I_2 \leq \frac{C_*^3}{2} \sum_{y=\bar{X}}^{\bar{Y}} t_y^2 |\tilde{f}_y^{(N)}|^2 \sum_{x=y-c_n\langle y \rangle^\gamma}^{y+c_n\langle y \rangle^\gamma} |a_{xy}| \frac{\langle x \rangle^k}{\langle y \rangle^k} \left| \frac{\langle x \rangle^k}{\langle y \rangle^k} - 1 \right|.$$

Choose X sufficiently large so that

$$\sum_{x=y-c_n\langle y \rangle^\gamma}^{y+c_n\langle y \rangle^\gamma} |a_{xy}| \frac{\langle x \rangle^k}{\langle y \rangle^k} \left| \frac{\langle x \rangle^k}{\langle y \rangle^k} - 1 \right| < \frac{\delta}{C_*^3}, \quad \text{for } |y| \geq \bar{X}.$$

As a result we conclude

$$I_2 \leq \frac{\delta}{2} \|T\tilde{f}^{(N)}\|_{\ell_2(\mathbb{Z})}^2. \quad (3.26)$$

Combining this with (3.25) we have

$$|\langle T\tilde{f}^{(N)}, [T, A^{(N)}]\tilde{f}^{(N)} \rangle_{\ell_2(\mathbb{Z})}| \leq \delta \sum_x t_x^2 |\tilde{f}_x^{(N)}|^2 = \delta \|T\tilde{f}^{(N)}\|_{\ell_2}^2.$$

Going back to (3.18) we obtain

$$\begin{aligned} \|T\tilde{f}^{(N)}\|_{\ell_2(\mathbb{Z})}^2 &\leq |\langle T\tilde{f}^{(N)}, Tg \rangle_{\ell_2(\mathbb{Z})}| + |\langle T\tilde{f}^{(N)}, [T, A^{(N)}]\tilde{f}^{(N)} \rangle_{\ell_2(\mathbb{Z})}| \\ &\leq |\langle T\tilde{f}^{(N)}, Tg \rangle_{\ell_2(\mathbb{Z})}| + \delta \|T\tilde{f}^{(N)}\|_{\ell_2(\mathbb{Z})}^2, \end{aligned}$$

therefore

$$\|T\tilde{f}^{(N)}\|_{\ell_2(\mathbb{Z})} \leq \frac{\|Tg\|_{\ell_2(\mathbb{Z})}}{1 - \delta}. \quad (3.27)$$

Now, let parameter Y , appearing in the definition of the operator T , tend to infinity. Since $\|Tg\|_{\ell_2(\mathbb{Z})}$ remains constant, starting with some sufficiently large Y (as $g \in c_0(\mathbb{Z})$) we infer that (3.27) implies (3.7). This ends the proof of (3.2) finishing also the proof of Theorem 2.1.

4. PROOF OF THEOREM 2.3

The argument used in the proof of Theorem 2.1 can be adapted to the present case, provided we are able to show that (3.2) holds for some fixed $k_0 > 2$. To prove this fact we repeat with no changes, except replacing c_n by n (maintaining the notation from the previous section), the calculations made between (3.15) and (3.21). Instead of (3.23) we write that for some constant $C > 0$

$$\left| \left(\frac{1 + (x + m)^2}{1 + x^2} \right)^{k_0/2} - 1 \right| \leq \frac{C}{\langle x \rangle}, \quad \forall x \in \mathbb{Z}, |m| \leq n. \quad (4.1)$$

Using the above together with (2.4) we conclude that for any $\epsilon > 0$ there exists X (appearing in the definition of operator T), depending on n , such that

$$\sum_{y=x-n}^{x+n} |a_{xy}| \left| \frac{\langle y \rangle^{k_0}}{\langle x \rangle^{k_0}} - 1 \right| \leq c_* + \epsilon, \quad \text{for } |x| \geq \bar{X} \quad (4.2)$$

and

$$\sum_{x=y-n}^{y+n} |a_{xy}| \frac{\langle x \rangle^{k_0}}{\langle y \rangle^{k_0}} \left| \frac{\langle x \rangle^{k_0}}{\langle y \rangle^{k_0}} - 1 \right| \leq c_* + \epsilon, \quad \text{for } |y| \geq \bar{X}.$$

Here $\bar{X} := X - n$. This leads to an estimate

$$I_1 + I_2 \leq \frac{1}{2}(c_* + \epsilon) [(n+1)^{k_0/2} + C_*^3] \|T\tilde{f}^{(N)}\|_{\ell_2(\mathbb{Z})}^2. \quad (4.3)$$

Choosing c_* and $\epsilon > 0$ in such a way that

$$\delta := \frac{1}{2}(c_* + \epsilon) [(n+1)^{k_0/2} + C_*^3] < 1$$

we can still claim (3.27). This allows us to conclude (3.2), which ends the proof of the theorem.

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